

GELFAND-SHILOV REGULARITY OF SG BOUNDARY VALUE PROBLEMS.

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ABSTRACT. We show that the solutions of SG elliptic boundary value problems defined on the complement of compact sets or on the half-space have some regularity in Gelfand-Shilov spaces. The results are obtained using classical results about Gevrey regularity of elliptic boundary value problems and Calderón projectors techniques adapted to the SG case. Recent developments about Gelfand-Shilov regularity of SG pseudo-differential operators on \mathbb{R}^n appear in an essential way.

Keywords: Pseudo-differential operators, elliptic boundary value problems.

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In this paper, we use pseudo-differential operators and Gelfand-Shilov spaces to obtain regularity results for elliptic boundary value problems on two different classes of unbounded open sets Ω : the complement of compact sets of \mathbb{R}^n - assuming some regularity on the boundary $\Gamma = \partial\Omega$ - and on the half-space. We are interested in the following boundary value problem:

$$\begin{aligned} Pu &= f, \text{ in } \Omega \\ B^j u &= g_j, \text{ on } \Gamma, j = 1, 2, \dots, r \end{aligned}$$

where the data f and g_j belong to appropriate Gelfand-Shilov spaces and P and B_j are differential operators, whose symbols belong to the SG class as in definition 9. The same class of boundary value problems were studied by C. Parenti, H. O. Cordes, A. K. Erkip and E. Schrohe [6, 8, 9, 10, 19]. Similar problems using Boutet de Monvel algebras were also studied by E. Schrohe, D. Kapanadze and B. W. Schulze [21, 13, 14, 22].

The class of SG differential and pseudo-differential operators were studied by many authors. Its definition goes back at least to C. Parenti [19] and H. O. Cordes [5]. They appear, for instance, in

Date: October 21, 2014.

2010 *Mathematics Subject Classification.* 35B65, 35S15, 35J40.

The author was supported by FAPESP (Processo n° 2012/18198-9).

quantum mechanics equations, in scattering theory problems, as in R. Melrose [16, Chapter 6], and in recent generalizations of fifth and seventh order of the KdV equation, as remarked by M. Cappiello, T. Gramchev and L. Rodino [2]. Elliptic boundary value problems on non-compact domains and manifolds using SG pseudo-differential operators were also studied by the already mentioned authors.

Recently M. Cappiello, T. Gramchev, F. Nicola and L. Rodino [3, 2, 18] have obtained, using pseudo-differential methods and Gelfand-Shilov spaces, more precise regularity results for linear and semi-linear SG elliptic problems in \mathbb{R}^n . They have applied these results to prove the exponential decay of solutions of traveling waves equations.

In our work, we study the same Gelfand-Shilov regularity, but for the class of SG elliptic boundary value problems studied by C. Parenti [19] and A. K. Erkip [8, 9]. Our main results essentially state that if the data of the SG elliptic boundary value problem are Gelfand-Shilov functions - or Gevrey on the bounded boundary of a set - then so is the solution. In order to do that, we first study and characterize the restrictions of Gelfand-Shilov functions to the classes of unbounded domains in which we are interested. The regularity results for boundary value problems on the complement of compact sets are then easily obtained. They are given by Theorem 21 of Section 3 and are a simple consequence of classical results that can be found in J. Lions and E. Magenes [15]. The regularity results on the half-space require a little more, as the border is not compact. First we have to investigate the behavior of the class of pseudo-differential operators defined by M. Cappiello, T. Gramchev and L. Rodino [3, 2, 18] on the half-space, obtaining a kind of transmission property in the sense of L. B. de Monvel [17] for Gelfand-Shilov functions. These results must be combined with Calderón projectors techniques [1, 20, 12, 23] in order to obtain the desired regularity. Our main regularity result for this case is given by Theorem 25 of Section 3.

1. THE GELFAND-SHILOV SPACE ON OPEN SETS OF \mathbb{R}^n .

In this section, we define the Gelfand-Shilov spaces on open sets Ω of \mathbb{R}^n . We show that, under certain assumptions, our definition coincides with the restriction of the usual Gelfand-Shilov functions to Ω . Let us first explain some notation used in this paper.

The open ball in \mathbb{R}^n of radius $r > 0$ and with center at the origin is denoted by $B_r(0)$. We denote by \mathbb{R}_+^n and \mathbb{R}_-^n the set of points $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ such that $x_n > 0$ and $x_n < 0$, respectively. The functions $r^\pm : \mathcal{D}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}_\pm^n)$ are just the restrictions of the distributions. The extension by 0 of a function defined in \mathbb{R}_-^n to \mathbb{R}^n or defined in \mathbb{R}_+^n to \mathbb{R}^n is denoted by $e^- : L^2(\mathbb{R}_-^n) \rightarrow L^2(\mathbb{R}^n)$ and $e^+ : L^2(\mathbb{R}_+^n) \rightarrow L^2(\mathbb{R}^n)$, respectively. The upper half-plane of \mathbb{C} is denoted by $\mathbb{H} := \{z \in \mathbb{C}; \text{Im}(z) \geq 0\}$ and its interior is denoted by $\overset{\circ}{\mathbb{H}} := \{z \in \mathbb{C}; \text{Im}(z) > 0\}$. We denote by $\langle \cdot \rangle : \mathbb{R}^n \rightarrow \mathbb{R}$ the function $\langle x \rangle := \sqrt{1 + |x|^2}$ and the set $\{0, 1, 2, \dots\}$ of non negative integers by \mathbb{N}_0 . The space $\mathcal{B}(\mathbb{C}^p, \mathbb{C}^q)$ is just the space of linear operators from \mathbb{C}^p to \mathbb{C}^q . The Gevrey functions in an open set Ω of order θ are denoted by $G^\theta(\Omega)$. The ones with compact support contained in Ω are denoted by $G_c^\theta(\Omega)$. As usual, $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of smooth functions whose derivatives are rapidly decreasing. If Ω is an open set of \mathbb{R}^n , then $\mathcal{S}(\Omega)$ denotes the set of restrictions of Schwartz functions to Ω . The main example is $\mathcal{S}(\mathbb{R}_+^n)$. For the Fourier transform, we use $\hat{u}(\xi) = \int e^{-ix\xi} u(x) dx$. We denote by $(\cdot, \cdot)_{L^2(\mathbb{R}^n)^{\oplus p}} : L^2(\mathbb{R}^n)^{\oplus p} \times L^2(\mathbb{R}^n)^{\oplus p} \rightarrow \mathbb{C}$ the

scalar product $(u, v)_{L^2(\mathbb{R}^n) \oplus p} = \sum_{j=1}^p \int u_j(x) \overline{v_j(x)} dx$, where $u = (u_1, \dots, u_p)$ and $v = (v_1, \dots, v_p)$. Finally, we use the multi-index notation, that is, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ and $D_x^\alpha := D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$, where $D_{x_j} = -i\partial_{x_j}$.

The next definition is a slight extension of the usual definition of Gelfand-Shilov functions which can be found in [11, 18]:

Definition 1. Let $\mu > 0$ and $\nu > 0$ be constants such that $\mu + \nu \geq 1$. Let $\Omega \subset \mathbb{R}^n$ be an open set. The Gelfand-Shilov space $\mathcal{S}_\nu^\mu(\Omega)$ is defined as the space of functions $u \in C^\infty(\Omega)$ for which there are constants $C > 0$ and $D > 0$ depending only on u such that

$$|x^\alpha \partial_x^\beta u(x)| \leq CD^{|\alpha|+|\beta|} (\alpha!)^\nu (\beta!)^\mu, \forall \alpha, \beta \in \mathbb{N}_0^n, \forall x \in \Omega.$$

For each constant $D > 0$, we define the subspace $\mathcal{S}_{\nu,D}^\mu(\Omega) \subset \mathcal{S}_\nu^\mu(\Omega)$ of the functions that satisfy the above estimate for the constant D . This is a Banach space whose norm is given by

$$\sup_{\alpha, \beta} \sup_{x \in \Omega} D^{-|\alpha|-|\beta|} (\alpha!)^{-\nu} (\beta!)^{-\mu} |x^\alpha \partial_x^\beta u(x)|.$$

The space $\mathcal{S}_\nu^\mu(\Omega)$ is endowed with the topology of inductive limit: $\mathcal{S}_\nu^\mu(\Omega) = \cup_{D>0} \mathcal{S}_{\nu,D}^\mu(\Omega)$. The continuous linear functionals on $\mathcal{S}_\nu^\mu(\Omega)$ are denoted by $\mathcal{S}_\nu^{\mu'}(\Omega)$.

It is clear from the definition that $G_c^\mu(\Omega) \subset \mathcal{S}_\nu^\mu(\Omega) \subset G^\mu(\Omega)$. Moreover, for bounded sets, $u \in \mathcal{S}_\nu^\mu(\Omega)$ if, and only if,

$$|\partial_x^\beta u(x)| \leq CD^{|\beta|} (\beta!)^\mu, \forall \beta \in \mathbb{N}_0^n, \forall x \in \Omega.$$

This means that, for $\mu \geq 1$, $u \in \mathcal{S}_\nu^\mu(\Omega)$ if, and only if, u is a Gevrey function of order μ , and the Gevrey estimates are uniform: the constants C and D hold for all Ω .

A function $u \in \mathcal{S}_\nu^\mu(\Omega)$ has an exponential decay of the form $|u(x)| \leq Ce^{-\epsilon|x|^{\frac{1}{\nu}}}$, for some $\epsilon > 0$. Therefore the study of the regularity in these spaces leads to a better understanding of the behavior of the solutions at the infinity as well as the Gevrey regularity of the solutions.

A simple and useful remark, which will be used in Section 3.2.1, is that $u \in \mathcal{S}_\nu^\mu(\Omega)$ iff for every $m \in \mathbb{R}$, there are constants $C > 0$ and $D > 0$ depending only on u and m such that

$$|x^\alpha \partial_x^\beta u(x)| \leq CD^{|\alpha|+|\beta|} (\alpha!)^\nu (\beta!)^\mu \langle x \rangle^m, \forall \alpha, \beta \in \mathbb{N}_0^n, \forall x \in \Omega.$$

In general, if $\Omega \neq \mathbb{R}^n$, not every function in $\mathcal{S}_\nu^\mu(\Omega)$ is necessarily the restriction of a function in $\mathcal{S}_\nu^\mu(\mathbb{R}^n)$, as the following example shows.

Example 2. Let $u : \mathbb{R}_+ \rightarrow \mathbb{C}$ be given by $u(x) = e^{-x}$. Hence $u \in \mathcal{S}_1^1(\mathbb{R}_+)$, but there is no function $v \in \mathcal{S}_1^1(\mathbb{R})$ such that $v(x) = u(x)$ for $x > 0$. In fact, if $v \in \mathcal{S}_1^1(\mathbb{R})$, then v extends to a holomorphic function in the strip $\{z \in \mathbb{C}, -T \leq \text{Im}(z) \leq T\}$ [18, Proposition 6.1.8.]. Hence $v(z) = e^{-z}$ everywhere. As $x \in \mathbb{R} \mapsto e^{-x}$ is not a function in $\mathcal{S}_1^1(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$, we obtain a contradiction.

For some situations, however, this is true. Let us study two situations: The half-space and the complement of a compact set. We start with the half-space situation.

Theorem 3. Let $f \in \mathcal{S}_\nu^\mu(\mathbb{R}_+^n)$, $\mu > 1$ and $\nu > 0$. Then there is a function $g \in \mathcal{S}_\nu^\mu(\mathbb{R}^n)$ such that $g(x) = f(x)$ for all $x \in \mathbb{R}_+^n$.

In order to prove this theorem, we use the functions and results of G. A. Džanašija [7]. Let us fix the constant $\mu > 1$ and define functions $\{a_k, k = 0, 1, 2, \dots\}$ and $\{b_k, k = 1, 2, 3, \dots\}$.

Definition 4. Let $D \geq 1$ and $r > 0$. Let us assume that $\frac{1}{2r} < \mu - 1$. We define functions $b_k : \mathbb{R} \rightarrow \mathbb{R}$, for $k \geq 1$, as

$$b_k(t) = \begin{cases} 0, & t \in]-\infty, -\sigma_k[\\ \exp\left(\frac{-k\sigma_k^{4r}}{t^{2r}(\sigma_k+t)^{2r}}\right), & t \in]-\sigma_k, 0[\\ 0, & t \in]0, \infty[\end{cases},$$

where $\sigma_k = D^{-1}k^{-(\mu-1)}$.

We define $a_k : \mathbb{R} \rightarrow \mathbb{R}$ in the following way: For $k \geq 1$, we define

$$a_k(t) = \begin{cases} \frac{\int_{-\infty}^t b_k(y)dy}{\int_{-\infty}^0 b_k(y)dy}, & t \in]-\infty, 0[\\ \frac{\int_{-\infty}^{-t} b_k(y)dy}{\int_{-\infty}^0 b_k(y)dy}, & t \in]0, \infty[\end{cases}.$$

For $k = 0$, we choose $a_0 = a_1$.

We note that for all $k \geq 0$, $\text{supp}(a_k) \subset [-1, 1]$, $a_k(0) = 1$ and $\left(\frac{d^l a_k}{dt^l}\right)(0) = 0$, for all $l > 0$. The next lemma gives the properties of the functions a_k that we need. For its proof, we refer to [7]. The constant $D \geq 1$ will be chosen along the proof of Theorem 3.

Lemma 5. [7] *There is a constant $T > 1$, depending only on $r > 0$, such that:*

(i) *If $k \leq \alpha_n$, then*

$$\left|\partial_{x_n}^{\alpha_n}(a_k(x_n)x_n^k)\right| \leq 2^{\alpha_n+1} \exp(ak) D^{-k} k^{-k(\mu-1)} T^{\alpha_n} D^{\alpha_n} \alpha_n^{\mu\alpha_n}.$$

(ii) *If $k > \alpha_n$, then*

$$\left|\partial_{x_n}^{\alpha_n}(a_k(x_n)x_n^k)\right| \leq 2^{\alpha_n+1} \exp(a(k+1)) D^{-k} k^{-k(\mu-1)} T^{\alpha_n} D^{\alpha_n} k^{\mu\alpha_n}.$$

In the above expressions $a := \frac{16^{2r}}{3^{2r}}$.

Using these functions and the above lemma, we can prove Theorem 8 by actually providing an extension of the function f .

Proof. (of Theorem 3) As $f \in \mathcal{S}_\nu^\mu(\mathbb{R}_+^n)$, there is a constant $B > 0$ such that

$$(1.1) \quad \left|x^\alpha (\partial_x^\beta f)(x)\right| \leq B^{|\alpha|+|\beta|+1} (\alpha!)^\nu (\beta!)^\mu, \quad \forall \alpha, \beta \in \mathbb{N}_0^n, \quad \forall x \in \mathbb{R}_+^n.$$

Let us choose and fix a constant $D \geq \max\{1, 2Be^{a+1}\}$, where D is the constant used in Definition 4.

We define a function $h \in C^\infty(\overline{\mathbb{R}_-^n})$ by

$$h(x) := \sum_{k=0}^{\infty} \frac{1}{k!} a_k(x_n) (\partial_{x_n}^k f)(x', 0) x_n^k,$$

where $(\partial_{x_n}^k f)(x', 0) := \lim_{x_n \rightarrow 0^+} (\partial_{x_n}^k f)(x', x_n)$.

We need to show that this series and its derivatives converge uniformly to a function that satisfies the Gelfand-Shilov estimates on \mathbb{R}_-^n . If this is the case, we see that

$$\left(\partial_{x'}^{\alpha'} \partial_{x_n}^{\alpha_n} h \right) (x', 0) = \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{x_n}^{\alpha_n} (a_k(x_n) x_n^k) \Big|_{x_n=0} \left(\partial_{x'}^{\alpha'} \partial_{x_n}^k f \right) (x', 0) = \left(\partial_{x'}^{\alpha'} \partial_{x_n}^{\alpha_n} f \right) (x', 0).$$

Hence the function $g : \mathbb{R}^n \rightarrow \mathbb{C}$ defined as

$$g(x) = \begin{cases} f(x), & x > 0 \\ h(x), & x \leq 0 \end{cases}$$

is such that $g \in \mathcal{S}_\nu^\mu(\mathbb{R}^n)$ and $g|_{\mathbb{R}_+^n} = f$. This is the function we are looking for.

Let $k \leq \alpha_n$. Then

$$\begin{aligned} \left| x'^{\beta'} x_n^{\beta_n} \partial_{x'}^{\alpha'} \partial_{x_n}^{\alpha_n} \left(\frac{1}{k!} a_k(x_n) (\partial_{x_n}^k f)(x', 0) x_n^k \right) \right| &= \frac{1}{k!} \left| x_n^{\beta_n} \partial_{x_n}^{\alpha_n} (a_k(x_n) x_n^k) \left(x'^{\beta'} \partial_{x'}^{\alpha'} \partial_{x_n}^k f \right) (x', 0) \right| \leq \\ &2^{\alpha_n+1} \exp(ak) D^{-k} k^{-k(\mu-1)} T^{\alpha_n} D^{\alpha_n} \alpha_n^{\mu \alpha_n} \frac{1}{k!} B^{|\alpha'|+|\beta'|+k+1} (\beta'!)^\nu (\alpha'!)^\mu (k!)^\mu \leq \\ &B^{|\alpha'|+|\beta'|+1} 2^{\alpha_n+1} T^{\alpha_n} D^{\alpha_n} (\beta'!)^\nu (\alpha'!)^\mu \alpha_n^{\mu \alpha_n} \left(\frac{\exp(a) B}{D} \right)^k. \end{aligned}$$

As $D \geq 2Be^a$, we obtain

$$\begin{aligned} \sum_{k=1}^{\alpha_n} \left| x'^{\beta'} x_n^{\beta_n} \partial_{x'}^{\alpha'} \partial_{x_n}^{\alpha_n} \left(\frac{1}{k!} a_k(x_n) (\partial_{x_n}^k f)(x', 0) x_n^k \right) \right| &\leq \\ B^{|\alpha'|+|\beta'|+1} 2^{\alpha_n+1} T^{\alpha_n} D^{\alpha_n} (\beta'!)^\nu (\alpha'!)^\mu \alpha_n^{\mu \alpha_n} \sum_{k=1}^{\alpha_n} \left(\frac{1}{2} \right)^k &\leq \tilde{C}_1 \tilde{D}_1^{|\alpha'|+|\beta'|} (\alpha'!)^\mu (\beta'!)^\nu, \end{aligned}$$

where $\tilde{C}_1 > 0$ and $\tilde{D}_1 > 0$ are constants that do not depend on α and β .

For $k > \alpha_n$, we have

$$\begin{aligned} \left| x'^{\beta'} x_n^{\beta_n} \partial_{x'}^{\alpha'} \partial_{x_n}^{\alpha_n} \left(\frac{1}{k!} a_k(x_n) (\partial_{x_n}^k f)(x', 0) x_n^k \right) \right| &= \left| \frac{1}{k!} x_n^{\beta_n} \partial_{x_n}^{\alpha_n} (a_k(x_n) x_n^k) \left(x'^{\beta'} \partial_{x'}^{\alpha'} \partial_{x_n}^k f \right) (x', 0) \right| \leq \\ &2^{\alpha_n+1} \exp[a(k+1)] D^{-k} k^{-k(\mu-1)} T^{\alpha_n} D^{\alpha_n} k^{\mu \alpha_n} \frac{1}{k!} B^{|\alpha'|+|\beta'|+k+1} (\beta'!)^\nu (\alpha'!)^\mu (k!)^\mu \leq \\ &B^{|\alpha'|+|\beta'|+1} 2^{\alpha_n+1} T^{\alpha_n} D^{\alpha_n} (\beta'!)^\nu (\alpha'!)^\mu k^{\mu \alpha_n} \exp(a) \left(\frac{\exp(a) B}{D} \right)^k. \end{aligned}$$

Using the inequality $e^{-a} \leq a^{-d} d^d e^{-d}$, for $a > 0$ and $d > 0$, we conclude that $k^{\mu \alpha_n} \leq e^k (\mu \alpha_n)^{\mu \alpha_n} e^{-\mu \alpha_n}$.

As $D \geq 2Be^{a+1}$, we obtain that

$$\begin{aligned} \sum_{k=\alpha_n+1}^{\infty} \left| \frac{1}{k!} x_n^{\beta_n} \partial_{x_n}^{\alpha_n} (a_k(x_n) x_n^k) \left(x'^{\beta'} \partial_{x'}^{\alpha'} \partial_{x_n}^k f \right) (x', 0) \right| &\leq \\ (\exp(a) 2B) B^{|\alpha'|+|\beta'|} (2TD\mu^\mu e^{-\mu})^{\alpha_n} (\beta'!)^\nu (\alpha'!)^\mu \alpha_n^{\mu \alpha_n} \sum_{k=\alpha_n+1}^{\infty} \left(\frac{\exp(a+1) B}{D} \right)^k &\leq \end{aligned}$$

$$\tilde{C}_2 \tilde{D}_2^{|\alpha|+|\beta'|} (\alpha!)^\mu (\beta')^\nu,$$

where $\tilde{C}_2 > 0$ and $\tilde{D}_2 > 0$ are constants that do not depend on α and β . \square

Remark 6. Precisely the same arguments can be used to obtain extensions of Gevrey functions of order $\mu > 1$: If $f \in C^\infty(\mathbb{R}_+^n)$ is a function such that for all bounded sets $B \subset \mathbb{R}_+^n$ - not only compacts - there are constants $C_B > 0$ and $D_B > 0$ such that

$$|\partial_x^\alpha f(x)| \leq C_B D_B^{|\alpha|} (\alpha!)^\mu, \forall x \in B,$$

then there is a Gevrey function \tilde{f} of order μ defined on \mathbb{R}^n such that $f = \tilde{f}|_{\mathbb{R}_+^n}$.

The second situation in which we are interested is in the complement of a bounded set. We need to be more precise about our assumptions.

Definition 7. Let U be a bounded open set. We say that its boundary $\Gamma = \partial U$ is a Gevrey $(n-1)$ -manifold of order Θ , U being locally on one side of Γ , if for every $y \in \Gamma$, there is a bounded open set $\mathcal{O} \subset \mathbb{R}^n$, $r_y > 0$ and a Gevrey diffeomorphism $\psi : \mathcal{O} \rightarrow B_{r_y}(0) \subset \mathbb{R}^n$ of order Θ such that $\psi(U \cap \mathcal{O}) = B_{r_y}(0) \cap \mathbb{R}_-^n$ and $\psi(\Gamma \cap \mathcal{O}) = \{x \in B_{r_y}(0); x_n = 0\}$. We also suppose that there exists a normal vector field ν on Γ , such that the functions ψ take ν to ∂_{x_n} .

Theorem 8. Let $\Omega = \mathbb{R}^n \setminus \overline{U}$, where U is a bounded open set, whose boundary $\Gamma = \partial U$ is a Gevrey $(n-1)$ -manifold of order Θ , U being locally on one side of Γ . If $f \in \mathcal{S}_\nu^\mu(\Omega)$, $\mu > 1$ and $\nu > 0$, then there is a function $g \in \mathcal{S}_\nu^{\tilde{\mu}}(\mathbb{R}^n)$, $\tilde{\mu} = \max\{\mu, \Theta\}$, such that $g(x) = f(x)$ for all $x \in \Omega$.

Proof. Let $\mathcal{O}_1, \dots, \mathcal{O}_N$ be bounded open sets such that $\Gamma = \cup_{j=1}^N \mathcal{O}_j$ and that there exist Gevrey diffeomorphisms of order Θ , $\psi_j : \mathcal{O}_j \rightarrow B_{r_j}(0) \subset \mathbb{R}^n$, as in Definition 7. Let $\mathcal{O}_{int} \subset U$ and $\mathcal{O}_{ext} \subset \Omega$ be open sets such that $\mathbb{R}^n = \mathcal{O}_{int} \cup \mathcal{O}_{ext} \cup \cup_{j=1}^N \mathcal{O}_j$. Let $\phi_{int}, \phi_{ext}, \phi_1, \dots, \phi_N$ be Gevrey functions of order Θ that form a partition of unity subordinate to the open cover $\{\mathcal{O}_{int}, \mathcal{O}_{ext}, \mathcal{O}_1, \dots, \mathcal{O}_N\}$.

Then $f \circ \psi_j^{-1} : B_{r_j}(0) \rightarrow \mathbb{C}$ is a function that satisfies, for all compact sets $K \subset B_{r_j}(0)$, the estimates

$$|\partial_x^\alpha (f \circ \psi_j^{-1})(x)| \leq C_K D_K^{|\alpha|} (\alpha!)^{\tilde{\mu}}, \forall x \in K \cap \mathbb{R}_+^n,$$

where $\tilde{\mu} = \max\{\mu, \Theta\}$.

Using Remark 6, we conclude that there is a Gevrey function $g_j : B_{r_j}(0) \rightarrow \mathbb{C}$ of order $\tilde{\mu}$ such that $g_j|_{B_{r_j}(0) \cap \mathbb{R}_+^n} = f \circ \psi_j^{-1}$.

Let us define $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{C}$ as

$$\tilde{f}(x) = f(x)\phi_{ext}(x) + \sum_{j=1}^n (g_j \circ \psi_j)(x)\phi_j(x).$$

Hence \tilde{f} extends f and it is a Gevrey function of order $\tilde{\mu}$ on \mathbb{R}^n . Let $R > 0$ be such that $U \subset B_R(0)$. Then

$$\left| x^\alpha \left(\partial_x^\beta \tilde{f} \right) (x) \right| \leq \begin{cases} C D^{|\alpha|+|\beta|} (\alpha!)^\nu (\beta!)^\mu, & \text{if } x \notin B_R(0) \\ \tilde{C} R^{|\alpha|} \tilde{D}^{|\beta|} (\beta!)^{\tilde{\mu}}, & \text{if } x \in \overline{B_R(0)} \end{cases}.$$

This implies that $\tilde{f} \in \mathcal{S}_\nu^{\tilde{\mu}}(\mathbb{R}^n)$. \square

2. SG PSEUDO-DIFFERENTIAL OPERATORS.

In this section, we recall the main properties of the SG calculus. A recent detailed exposition can be found in F. Nicola and L. Rodino [18]. The calculus with symbols in $S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ were originally, as far as we know, defined and studied by Capiello and Rodino in [3].

Definition 9. (*SG symbols*) Let m_1 and m_2 belong to \mathbb{R} . We denote by $S^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ the set of all functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying, for all $\alpha, \beta \in \mathbb{N}_0^n$, the estimates

$$|D_x^\beta D_\xi^\alpha a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m_1 - |\alpha|} \langle x \rangle^{m_2 - |\beta|}, \quad \forall (x, \xi) \in \mathbb{R}^{2n},$$

where $C_{\alpha\beta}$ is a constant that depends on a , α and β . These functions are called SG symbols of class $(m_1, m_2) \in \mathbb{R}^2$.

Similarly, we denote by $S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$, where μ and ν are real numbers such that $\mu \geq 1$ and $\nu \geq 1$, the subspace of $S^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ defined as follows: $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ if there are constants $C > 0$ and $D > 0$ depending only on a such that $C_{\alpha\beta} = CD^{|\alpha|+|\beta|} (\alpha!)^\mu (\beta!)^\nu$.

We denote by $S^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$ and $S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$ the classes of functions $a : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q)$, where each entry of the matrix belongs to $S^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ and $S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$, respectively.

It is clear that we could do the same definitions also for $(x, \xi) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. We denote these spaces by $S^{m_1, m_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, $S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and so on.

We are mostly interested in elliptic symbols.

Definition 10. Let $a \in S^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$, $(m_1, m_2) \in \mathbb{R}^2$. We say that the symbol a is left (right) elliptic if there are constants $R > 0$ and $C > 0$ such that if $|(x, \xi)| \geq R$, then $a(x, \xi)$ has a left (right) inverse $(x, \xi) \mapsto b(x, \xi)$ such that $\|b(x, \xi)\|_{\mathcal{B}(\mathbb{C}^q, \mathbb{C}^p)} \leq C \langle x \rangle^{-m_2} \langle \xi \rangle^{-m_1}$. In particular, $q \geq p$ ($q \leq p$). A symbol is elliptic iff it is left and right elliptic. In this case $p = q$ and $\|a(x, \xi)^{-1}\|_{\mathcal{B}(\mathbb{C}^q, \mathbb{C}^p)} \leq C \langle x \rangle^{-m_2} \langle \xi \rangle^{-m_1}$, if $|(x, \xi)| \geq R$.

The same ellipticity definition holds for $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$.

For a symbol $a \in S^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$, $(m_1, m_2) \in \mathbb{R}^2$, left ellipticity is equivalent to the ellipticity of the symbol $a^*a \in S^{2m_1, 2m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^p))$. Hence, if b is a parametrix of a^*a , then ba^* is a left parametrix of a . The analogous result holds for right elliptic symbols.

As usual we define pseudo-differential and regularizing operators associated with these symbols.

Definition 11. For each symbol a in the classes $S^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ and $S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$, we define a pseudo-differential operator $A = op(a) : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ by the formula:

$$Au(x) = \frac{1}{(2\pi)^n} \int e^{ix\xi} a(x, \xi) \hat{u}(\xi) d\xi,$$

where $\hat{u}(\xi) = \int e^{-ix\xi} u(x) dx$. For matrix symbols $a = (a_{ij})$ that belong to $S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$ and $S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$, we define $A = op(a) : \mathcal{S}(\mathbb{R}^n)^{\oplus p} \rightarrow \mathcal{S}(\mathbb{R}^n)^{\oplus q}$ by:

$$(2.1) \quad (Au)_k(x) = \sum_{j=1}^p \frac{1}{(2\pi)^n} \int e^{ix\xi} a_{kj}(x, \xi) \hat{u}_j(\xi) d\xi.$$

Definition 12. Let $\theta > 1$. A linear continuous operator from $\mathcal{S}_\theta^\theta(\mathbb{R}^n)^{\oplus p}$ to $\mathcal{S}_\theta^\theta(\mathbb{R}^n)^{\oplus q}$ is said to be θ -regularizing operator if it extends to a linear continuous map from $\mathcal{S}_\theta^{\theta'}(\mathbb{R}^n)^{\oplus p}$ to $\mathcal{S}_\theta^\theta(\mathbb{R}^n)^{\oplus q}$.

Operators whose kernel is a $q \times p$ matrix with entries in $\mathcal{S}_\theta^\theta(\mathbb{R}^n \times \mathbb{R}^n)$, also denoted by $\mathcal{S}_\theta^\theta(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$, are θ -regularizing operators.

Let us now give the properties of the pseudo-differential operators with symbols in $S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$, which we shall need. They can be found in [18, Chapter 6].

Proposition 13. Let $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$, $(m_1, m_2) \in \mathbb{R}^2$, μ and ν be real numbers such that $\mu \geq 1$, $\nu \geq 1$, p and q integers such that $p \geq 1$ and $q \geq 1$. For every $\theta \geq \max\{\mu, \nu\}$, the operator defined in 2.1 is a linear continuous operator from $\mathcal{S}_\theta^{\theta'}(\mathbb{R}^n)^{\oplus p}$ to $\mathcal{S}_\theta^\theta(\mathbb{R}^n)^{\oplus q}$.

Let us define $H^{s_1, s_2}(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n); \langle x \rangle^{s_2} \langle D \rangle^{s_1} u \in L^2(\mathbb{R}^n)\}$. Then the above operator extends to a continuous operator from $H^{s_1, s_2}(\mathbb{R}^n)^{\oplus p}$ to $H^{s_1 - m_1, s_2 - m_2}(\mathbb{R}^n)^{\oplus q}$.

Definition 14. Let us denote by $FS_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ the space of all formal sums $\sum_{j \geq 0} a_j$, where the functions $a_j \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the following condition: There are constants B, C and $D > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a_j(x, \xi)| \leq CD^{|\alpha|+|\beta|+2j} (\alpha!)^\mu (\beta!)^\nu (j!)^{\mu+\nu-1} \langle x \rangle^{m_2-j-|\beta|} \langle \xi \rangle^{m_1-j-|\alpha|},$$

whenever $\langle x \rangle \geq Bj^{\mu+\nu-1}$ or $\langle \xi \rangle \geq Bj^{\mu+\nu-1}$. In the same way, by $FS_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$, we denote the space of all formal sums $\sum_{j \geq 0} a_j$, where the functions $a_j \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$ are such that the formal sum of each of its entry belongs to $FS_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$.

Definition 15. We say that $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ has the asymptotic expansion $a \sim \sum_{j \geq 0} a_j$, if $\sum_{j \geq 0} a_j \in FS_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ and if there exist constants B, C and $D > 0$ such that

$$\left| \partial_\xi^\alpha \partial_x^\beta \left(a - \sum_{j=0}^{N-1} a_j \right) (x, \xi) \right| \leq CD^{|\alpha|+|\beta|+2N} (\alpha!)^\mu (\beta!)^\nu (N!)^{\mu+\nu-1} \langle x \rangle^{m_2-N-|\beta|} \langle \xi \rangle^{m_1-N-|\alpha|},$$

whenever $\langle x \rangle \geq BN^{\mu+\nu-1}$ or $\langle \xi \rangle \geq BN^{\mu+\nu-1}$. The analogous holds for matricial symbols.

Proposition 16. Let $\sum_{j \geq 0} a_j \in FS_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$. Then there exists a symbol $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$ such that $a \sim \sum_{j \geq 0} a_j$. Moreover if a and b are two functions that belong to $S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$ and are such that $a \sim \sum_{j \geq 0} a_j$ and $b \sim \sum_{j \geq 0} a_j$, then $a - b \in \mathcal{S}_\theta^\theta(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$, that is, $op(a) - op(b)$ is a θ -regularizing operator for any $\theta \geq \mu + \nu - 1$.

Proposition 17. Let $\mu > 1$ and $\nu > 1$. Let us consider the symbols $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$, $(m_1, m_2) \in \mathbb{R}^2$, and $b \in S_{\mu'\nu'}^{m'_1, m'_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^r, \mathbb{C}^p))$, $(m'_1, m'_2) \in \mathbb{R}^2$. Then, for $\theta \geq \mu + \nu - 1$:

(i) There is a symbol $c \in S_{\mu\nu}^{m_1+m'_1, m_2+m'_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^r, \mathbb{C}^q))$, also denoted by $a \sharp b$, and a θ -regularizing operator $R : \mathcal{S}_\theta^{\theta'}(\mathbb{R}^n)^{\oplus r} \rightarrow \mathcal{S}_\theta^\theta(\mathbb{R}^n)^{\oplus q}$, where $\theta \geq \mu + \nu - 1$, such that $op(c) = op(a) \circ op(b) + R$. Its

symbol has the following asymptotic expansion

$$a \# b \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} a D_x^{\alpha} b.$$

(ii) There is a symbol $c \in S^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^q, \mathbb{C}^p))$, also denoted a^* , and a θ -regularizing operator $R : \mathcal{S}_{\theta}^{\theta'}(\mathbb{R}^n)^{\oplus q} \rightarrow \mathcal{S}_{\theta}^{\theta}(\mathbb{R}^n)^{\oplus p}$, where $\theta \geq \mu + \nu - 1$, such that, for all $u, v \in \mathcal{S}(\mathbb{R}^n)$, the following equality holds

$$(op(a)u, v)_{L^2(\mathbb{R}^n)^{\oplus q}} = (u, op(a^*)v)_{L^2(\mathbb{R}^n)^{\oplus p}} + (u, Rv)_{L^2(\mathbb{R}^n)^{\oplus p}}.$$

The symbol a^* has the following asymptotic expansion

$$a^* \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} D_x^{\alpha} \bar{a}.$$

We finally state the following important regularity result:

Theorem 18. Let $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^p, \mathbb{C}^q))$, $\mu > 1$, $\nu > 1$, be a left (right) elliptic symbol. Then there is a symbol called left (right) parametrix $b \in S_{\mu\nu}^{-m_1, -m_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{C}^q, \mathbb{C}^p))$ such that $op(b)op(a) = I + R$ ($op(a)op(b) = I + R$), where $R : \mathcal{S}_{\theta}^{\theta'}(\mathbb{R}^n)^{\oplus p} \rightarrow \mathcal{S}_{\theta}^{\theta}(\mathbb{R}^n)^{\oplus p}$ ($R : \mathcal{S}_{\theta}^{\theta'}(\mathbb{R}^n)^{\oplus q} \rightarrow \mathcal{S}_{\theta}^{\theta}(\mathbb{R}^n)^{\oplus q}$) is a θ -regularizing operator and θ is any number that satisfies $\theta \geq \mu + \nu - 1$.

In particular, if a is a left elliptic symbol, $u \in \mathcal{S}_{\theta}^{\theta'}(\mathbb{R}^n)^{\oplus p}$ and $op(a)u \in \mathcal{S}_{\theta}^{\theta}(\mathbb{R}^n)^{\oplus q}$ for $\theta \geq \mu + \nu - 1$, then $u \in \mathcal{S}_{\theta}^{\theta}(\mathbb{R}^n)^{\oplus p}$.

Note that if a is a left elliptic symbol of a differential operator, then $\mu = 1$. Hence if $u \in \mathcal{S}_{\theta}^{\theta'}(\mathbb{R}^n)^{\oplus p}$ is such that $op(a)u \in \mathcal{S}_{\theta}^{\theta}(\mathbb{R}^n)^{\oplus q}$ for $\theta > \nu$, then $u \in \mathcal{S}_{\theta}^{\theta}(\mathbb{R}^n)^{\oplus p}$.

3. REGULARITY RESULTS

3.1. Elliptic SG boundary value problems on the complement of compact sets. In this section we prove regularity in Gelfand-Shilov spaces of solutions of SG boundary value problems on the complement of compact sets, as introduced by C. Parenti [19, Section 3].

Let U be a bounded open set such that its boundary $\Gamma = \partial U$ is a Gevrey $(n - 1)$ -manifold of order $\Theta > 1$, U being locally on one side of Γ . Let $\Omega = \mathbb{R}^n \setminus \overline{U}$. We consider the following boundary value problem:

$$\begin{aligned} Pu &= f, \text{ in } \Omega \\ B^j u &= g_j, \text{ on } \Gamma, j = 1, 2, \dots, r \end{aligned}$$

where:

a) $P(x, D) = \sum_{|\alpha| \leq m_1} a_{\alpha}(x) D_x^{\alpha}$ is a differential operator on \mathbb{R}^n and $m_1 = 2r$. We assume that the functions $a_{\alpha} \in C^{\infty}(\mathbb{R}^n)$ satisfy the following estimates for some $\nu \geq 1$

$$|\partial_x^{\beta} a_{\alpha}(x)| \leq CD^{|\beta|} (\beta!)^{\nu} \langle x \rangle^{m_2 - |\beta|}, \forall x \in \mathbb{R}^n.$$

Hence the function $a \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ given by $a(x, \xi) = \sum_{|\alpha| \leq m_1} a_{\alpha}(x) \xi^{\alpha}$ belongs to $S_{1\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$

b) For each $j = 1, \dots, r$, we associate an integer number $0 \leq m_{1j} \leq m_1 - 1$ and $m_{2j} \in \mathbb{R}$. Let $B = (B_{j,k})$, with $1 \leq j \leq r$ and $0 \leq k \leq m_1 - 1$, be a matrix, where $B_{j,k}$ is a differential operator of order $m_{1j} - k$ on Γ whose coefficients are Gevrey functions of order Θ . We assume also that $B_{j,k} = 0$, if $k > m_{1j}$. For each $u \in \mathcal{S}(\Omega)$, we define $\gamma(u) := (\gamma_0(u), \dots, \gamma_{m_1-1}(u))$, where γ_j is defined using the charts of Definition 7: $\gamma_j(u) \circ \psi^{-1}(x) := \lim_{x_n \rightarrow 0^+} D_{x_n}^j (u \circ \psi^{-1})(x, x_n)$. The derivative D_{x_n} is the one associated with the field ν , again as in Definition 7. The operator $B^j : \mathcal{S}(\Omega) \rightarrow C^\infty(\Gamma)^{\oplus r}$ is defined as $B^j u = \sum_{k=0}^{m_{1j}-1} B_{j,k}(x', D') \gamma_k(u)$, $j = 1, 2, \dots, r$.

c) The symbol a is SG-elliptic, properly elliptic in the classical sense and the boundary value problem satisfies the usual Lopatinski-Shapiro condition at the boundary. We recall the definition below.

d) There is a $\theta > \nu$ and $\theta \geq \Theta$ such that the functions g_j are Gevrey functions of order θ in Γ and $f \in \mathcal{S}_\theta^\theta(\Omega)$.

Definition 19. 1) Let us define $a_{(m_1)}(x, \xi) = \sum_{|\alpha|=m_1} a_\alpha(x) \xi^\alpha$. We say that the function a is properly elliptic in the classical sense if it is elliptic - it is non zero if $\xi \neq 0$ - and for all $x \in \Gamma$, ξ_1 and ξ_2 linearly independent vectors in \mathbb{R}^n , the polynomial $z \in \mathbb{C} \mapsto a_{(m_1)}(x, \xi_1 + z\xi_2)$ has exactly $r = \frac{m_1}{2}$ roots with positive imaginary part - and, hence, r roots with negative imaginary part. We denote these roots by $\tau_1(x, \xi_1, \xi_2), \dots, \tau_r(x, \xi_1, \xi_2)$ and we set $a_{(m_1)}^+(x, \xi_1, \xi_2)(z) := \prod_{j=1}^r (z - \tau_j(x, \xi_1, \xi_2))$.

2) Let us write $B^j u(x) = \sum_{|\alpha| \leq m_{1j}} b_\alpha^j(x) D^\alpha$, for $x \in \Gamma$. We define the polynomials

$$b_{(m_{1j})}^j(x, \xi, \xi')(z) := \sum_{|\alpha|=m_{1j}} b_\alpha^j(x) (\xi + z\xi')^\alpha,$$

where $x \in \Gamma$, ξ is tangent to Γ and ξ' is normal to Γ . The boundary value problem satisfies the classical Lopatinski-Shapiro (or covering) condition if, for all $x \in \Gamma$, $\xi \neq 0$ tangent to Γ and $\xi' \neq 0$ normal to Γ , the polynomials $z \in \mathbb{C} \mapsto b_{(m_{1j})}^j(x, \xi, \xi')(z)$ are linearly independent modulo $z \in \mathbb{C} \mapsto a_{(m_1)}^+(x, \xi, \xi')(z)$.

Let us recall the classical result about Gevrey regularity of elliptic boundary value problems:

Theorem 20. (Theorem 1.3 of J. Lions and E. Magenes [15]) Let $0 < \rho_0 < 1$ be a fixed constant and let us consider the following boundary value problem in $B_{\rho_0}(0) \cap \mathbb{R}_+^n$:

$$\begin{aligned} Pu &= f, \quad \text{in } B_{\rho_0}(0) \cap \mathbb{R}_+^n \\ B^j u &= g_j, \quad \text{on } B_{\rho_0}(0) \cap \partial(\mathbb{R}_+^n), \quad j = 1, 2, \dots, r \end{aligned}$$

where:

- 1) $P(x, D)u = \sum_{|\alpha| \leq m_1} a_\alpha(x) D^\alpha u(x)$, $m_1 = 2r$, is a properly elliptic operator on $B_{\rho_0}(0) \cap \partial(\mathbb{R}_+^n)$ and a_α are restrictions of Gevrey functions defined on \mathbb{R}^n of order $\beta > 1$ to $B_{\rho_0}(0) \cap \mathbb{R}_+^n$.
- 2) $B^j u(x') = \sum_{|\alpha| \leq m_{1j}} b_{j\alpha}(x') D^\alpha u(x', 0)$ are r boundary operators and $b_{j\alpha}$ are Gevrey functions of order β defined on $B_{\rho_0}(0) \cap \partial(\mathbb{R}_+^n)$.

If $u \in C^\infty(\overline{B_{\rho_0}(0) \cap \mathbb{R}_+^n})$, f and g_j , $1 \leq j \leq r$, are Gevrey functions of order β and the above boundary value problem satisfies the classical Lopatinski-Shapiro condition, then there is a $\rho' < \rho_0$, such that $u|_{B_{\rho'}(0) \cap \mathbb{R}_+^n}$ is the restriction of a Gevrey function of order β defined on \mathbb{R}^n to $B_{\rho'}(0) \cap \mathbb{R}_+^n$.

We finally prove our main result on the complement of compact sets:

Theorem 21. (*Main Theorem on the complement of compact sets*) Let $u \in H^{s_1, s_2}(\Omega)$, the space of restrictions of distributions in $H^{s_1, s_2}(\mathbb{R}^n)$ to Ω , $s_1 \geq m_1$, be a solution of

$$\begin{aligned} Pu &= f, \text{ in } \Omega \\ B^j u &= g_j, \text{ on } \Gamma, j = 1, 2, \dots, r \end{aligned}$$

where the boundary value problem satisfies the conditions a, b, c and d. Then $u \in \mathcal{S}_\theta^\theta(\Omega)$.

Proof. As f belongs to $\mathcal{S}(\Omega) = \cap_{(s_1, s_2) \in \mathbb{R}^2} H^{s_1, s_2}(\Omega)$ and g_j belongs to $C^\infty(\Gamma) = \cap_{s \in \mathbb{R}} H^s(\Gamma)$, for all j , we conclude that $u \in \mathcal{S}(\Omega)$, according to C. Parenti [19, Section 3]. For each $x \in \Gamma$, there exists a neighborhood $\mathcal{O}_x \subset \mathbb{R}^n$ of x , $r_x > 0$ and a Gevrey diffeomorphism $\psi_x : \mathcal{O}_x \rightarrow B_{r_x}(0)$ of order Θ , as in Definition 7. Due to Theorem 20, there is a ball $B_{\rho_x}(0) \subset B_{r_x}(0)$, $\rho_x < 1$, such that $u \circ \psi_x^{-1} : B_{\rho_x}(0) \cap \mathbb{R}_+^n \rightarrow \mathbb{C}$ is the restriction of a Gevrey function $\tilde{u}_x : \mathbb{R}^n \rightarrow \mathbb{C}$ of order θ . Let us choose x_1, \dots, x_N such that $\Gamma \subset \cup_{j=1}^N \psi_{x_j}^{-1}(B_{\rho_{x_j}}(0))$. Let us also choose Gevrey functions with compact support χ_1, \dots, χ_N of order θ taking values on $[0, 1]$ and such that $\sum_{j=1}^N \chi_j = 1$ in a neighborhood of Γ and that $\text{supp}(\chi_j) \subset \psi_{x_j}^{-1}(B_{\rho_{x_j}}(0))$. Let us define

$$\tilde{u}(x) = \sum_{j=1}^N \chi_j(x) \tilde{u}_{x_j} \circ \psi_{x_j}(x).$$

This is a Gevrey function of order θ . Moreover, there is a bounded neighborhood of Γ , $V \subset \mathbb{R}^n$, such that $\tilde{u}|_{V \cap \Omega} = u|_{V \cap \Omega}$. Let us now choose a Gevrey function of order θ , $\chi : \mathbb{R}^n \rightarrow \mathbb{C}$, such that $\chi(x) = 1$ in a neighborhood of the complement of $V \cup \Omega$ and $\chi(x) = 0$ in a neighborhood of $\overline{\Omega}$. Hence, on Ω , we have $P(\chi u) = f - P((1 - \chi)\tilde{u})$. However $f - P((1 - \chi)\tilde{u})$ is zero in a neighborhood of Γ and $P((1 - \chi)\tilde{u})$ is a Gevrey function of order θ with compact support. This means that $f - P((1 - \chi)\tilde{u})$ can be extended to a Gelfand-Shilov function in $\mathcal{S}_\theta^\theta(\mathbb{R}^n)$ - we only have to extend by zero on \overline{U} . As $\theta > \nu$, we conclude, using Theorem 18, that $\chi u \in \mathcal{S}_\theta^\theta(\mathbb{R}^n)$. As u is the restriction of the function $\chi u + (1 - \chi)\tilde{u}$ to Ω , we conclude that $u \in \mathcal{S}_\theta^\theta(\Omega)$. \square

3.2. Elliptic SG boundary value problems on the half-space. In this section we prove regularity in Gelfand-Shilov spaces of solutions of SG boundary value problems on the half-space, as introduced by H. O. Cordes and A. K. Erkip [6, 8, 9].

We consider the following boundary value problem:

$$\begin{aligned} Pu &= f, \text{ in } \mathbb{R}_+^n \\ B^j u &= g_j, \text{ on } \mathbb{R}^{n-1}, j = 1, 2, \dots, r \end{aligned}$$

where:

a) $P(x, D) = \sum_{|\alpha| \leq m_1} a_\alpha(x) D_x^\alpha$ is a differential operator on \mathbb{R}^n and $m_1 = 2r$. We assume that the functions $a_\alpha \in C^\infty(\mathbb{R}^n)$ satisfy the following estimates for $\nu \geq 1$

$$|\partial_x^\beta a_\alpha(x)| \leq CD^{|\beta|} (\beta!)^\nu \langle x \rangle^{m_2 - |\beta|}.$$

Hence the function $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ given by $a(x, \xi) = \sum_{|\alpha| \leq m_1} a_\alpha(x) \xi^\alpha$ belongs to $S_{1\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$

b) For each $j = 1, \dots, r$, we associate an integer $0 \leq m_{1j} \leq m_1 - 1$ and $m_{2j} \in \mathbb{R}$. Let $B = (B_{j,k})$, with $1 \leq j \leq r$ and $0 \leq k \leq m_1 - 1$ be a matrix, where $B_{j,k}$ is a pseudo-differential operator, whose symbol belongs to $S_{1\nu}^{m_{1j}-k, m_{2j}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, $j = 1, \dots, r$. We assume also that $B_{j,k} = 0$, if $k > m_{1j}$. For each $u \in \mathcal{S}(\mathbb{R}_+^n)$, we define $\gamma(u) := (\gamma_0(u), \dots, \gamma_{m_1-1}(u))$, where $\gamma_j(u) := \lim_{x_n \rightarrow 0^+} (D_{x_n}^j u)(x, x_n)$. The operator $B^j : \mathcal{S}(\mathbb{R}_+^n) \rightarrow \mathcal{S}(\mathbb{R}^{n-1})^{\oplus r}$ is defined as

$$B^j u = \sum_{k=0}^{m_1-1} B_{j,k}(x', D') \gamma_k(u), \quad j = 1, 2, \dots, r.$$

c) The boundary value problem is SG elliptic, as defined below.

d) There is a $\theta > \nu$ such that $g_j \in \mathcal{S}_\theta^\theta(\mathbb{R}^{n-1})$, $\forall j$, and $f \in \mathcal{S}_\theta^\theta(\mathbb{R}_+^n)$.

Definition 22. We say that the above boundary value problem is SG elliptic if it satisfies the following conditions [8, 9]:

1) Let us define the function $a_{(x', \xi')}(z) := \langle x' \rangle^{-m_2} \langle \xi' \rangle^{-m_1} a(x', 0, \xi', \langle \xi' \rangle z)$. The function a is SG-properly elliptic: it is SG-elliptic as in Definition 10 and there is an $R > 0$ such that, for $|(x, \xi)| \geq R$, the polynomial $z \in \mathbb{C} \mapsto a_{(x', \xi')}(z)$ has exactly r roots with positive imaginary part - and r roots with negative imaginary part. We denote these roots by $\tau_1(x', \xi')$, ..., $\tau_r(x', \xi')$ and we set $a_{(x', \xi')}^+(z) := \prod_{j=1}^r (z - \tau_j(x', \xi'))$.

2) Let us define the polynomials $b_{(x', \xi')}^j(z) := \sum_{k=0}^{m_1-1} B_{j,k}(x', \xi') \langle x' \rangle^{-m_{2j}} \langle \xi' \rangle^{k-m_{1j}} z^k$. The boundary value problem satisfies the SG-Lopatinski-Shapiro (or covering) condition: there exists $R > 0$ such that if $|(x', \xi')| \geq R$, then the polynomials $b_{(x', \xi')}^j(z)$ are uniformly and linearly independent modulo $a_{(x', \xi')}^+(z)$. This means that $b_{(x', \xi')}^j(z) = \tilde{b}_{(x', \xi')}^j(z) \bmod a_{(x', \xi')}^+(z)$, where $\tilde{b}_{(x', \xi')}^j(z) = \sum_{k=0}^{r-1} \tilde{b}_{(x', \xi')}^{j,k} z^k$, and for $|(x', \xi')| \geq R$, there exists a constant $C > 0$, independent of (x', ξ') , such that $\left| \det \left(\tilde{b}_{(x', \xi')}^{j,k}(z) \right) \right| \geq C$.

Example 23. Let us consider the Dirichlet problem:

$$\begin{aligned} Pu &= f, \text{ in } \mathbb{R}_+^n \\ \langle x' \rangle^{m_{2j}} \gamma^{j-1}(u) &= g_j, \text{ on } \mathbb{R}^{n-1}, j = 1, 2, \dots, r \end{aligned}$$

where $P(x, D) = \sum_{|\alpha| \leq m_1} a_\alpha(x) D^\alpha$, $m_1 = 2r$, is a SG-properly elliptic differential operator. Then $b_{(x', \xi')}^j(z) = \tilde{b}_{(x', \xi')}^j(z) = z^j$ and $\tilde{b}_{(x', \xi')}^{j,k} = \delta_{jk}$. This clearly satisfies the SG-Lopatinski-Shapiro condition.

Remark 24. It is very important to note, as in [8, 9], that, if $a \in S^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ is an SG elliptic symbol of a differential operator, then there is a constant $D > 0$ that does not depend on (x, ξ') , such that the roots of the polynomial $z \in \mathbb{C} \mapsto a(x, \xi', z)$ satisfy $|z| \leq D \langle \xi' \rangle$, for all (x, ξ') . If we write $a(x, \xi', z) = \sum_{j=0}^{m_1} P_j(x, \xi') z^j$, then $P_j \in S^{m_1-j, m_2}(\mathbb{R}^n \times \mathbb{R}^{n-1})$ and P_{m_1} only depends on x . By the ellipticity assumption, $|P_{m_1}(x)| \geq C \langle x \rangle^{m_1}$ for a constant $C > 0$ that does not depend on x . The result follows then easily from the simple fact that the roots of a polynomial $P(z) = \sum_{j=0}^N P_j z^j$ belong to the ball of radius $\max_j \left\{ \left(N \left| \frac{P_j}{P_N} \right| \right)^{\frac{1}{N-j}} \right\}$.

The Theorem below is our main result on the half-space.

Theorem 25. (Main Theorem on the half-space) Let $u \in H^{s_1, s_2}(\mathbb{R}_+^n)$, the space of restrictions of distributions in $H^{s_1, s_2}(\mathbb{R}^n)$ to \mathbb{R}_+^n , $s_1 \geq m_1$, be a solution of

$$(3.1) \quad \begin{aligned} Pu &= f, \text{ in } \mathbb{R}_+^n \\ B^j u &= g_j, \text{ on } \mathbb{R}^{n-1}, j = 1, 2, \dots, r \end{aligned}$$

where the boundary value problem satisfies the conditions a , b , c and d . Then $u \in \mathcal{S}_\theta^\theta(\mathbb{R}_+^n)$.

In order to prove that, we use a very classical pseudo-differential approach. We follow closely the ideas of L. Hörmander [12] and the presentation of J. Chazarain and A. Piriou [4]. First it is necessary to study the behaviour of a subclass of SG-pseudo-differential operators near the boundary.

3.2.1. *The behaviour of SG pseudo-differential operators near the boundary.* In this section we will always assume that $\mu > 1$ and $\nu > 1$.

We are mainly concerned with the behaviour of parametrices of SG elliptic differential operators. Let us start studying this case in order to clarify our assumptions.

Let $a \in S_{1\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ be an SG elliptic differential symbol. Hence there are constants $C > 0$, $R > 0$ and $r > 0$ such that $|a(x, \xi)| \geq C \langle x \rangle^{m_2} \langle \xi \rangle^{m_1}$, for all $|(x, \xi)| \geq R$, and such that all the roots of the polynomial $z \in \mathbb{C} \mapsto a(x, \xi', z)$ lie in some ball of radius $r \langle \xi' \rangle$.

Using the definition of $S_{1\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$, it is clear that, if $|(x, \xi)| \geq R$, then there exist constants $C > 0$ and $D > 0$ such that

$$\left| \partial_x^\beta \partial_\xi^\alpha \left(\frac{1}{a(x, \xi)} \right) \right| \leq CD^{|\alpha|+|\beta|} \alpha! (\beta!)^\nu \langle x \rangle^{-m_2-|\beta|} \langle \xi \rangle^{-m_1-|\alpha|}.$$

If $\tau_1(x, \xi')$, ..., $\tau_{m_1}(x, \xi')$ are the roots of the polynomial $z \in \mathbb{C} \mapsto a(x, \xi', z)$, then $a(x, \xi', z) = \tilde{a}(x, \xi') \prod_{j=1}^{m_1} (z - \tau_j(x, \xi'))$, for some function \tilde{a} . Using the SG ellipticity property for $a(x, \xi', 0)$ and the fact that $|\tau_j(x, \xi')| \leq r \langle \xi' \rangle$ for all j , we conclude that there exists a constant $\tilde{C} > 0$ such that, for all $|(x, \xi')| \geq R$, $|\tilde{a}(x, \xi')| \geq \tilde{C} \langle x \rangle^{m_2}$.

Let $\tilde{R} \geq r$. Then if $z = \tilde{R} \langle \xi' \rangle e^{i\theta}$, then $|z - \tau_j(x, \xi')| \geq (\tilde{R} - r) \langle \xi' \rangle$, for all j . Hence there exists a constant $C > 0$ such that

$$\left| a(x, \xi', \tilde{R} \langle \xi' \rangle e^{i\theta}) \right| \geq C \langle x \rangle^{m_2} \langle \xi' \rangle^{m_1},$$

for all $|(x, \xi')| \geq R$. As $a(x, \xi', \tilde{R} \langle \xi' \rangle e^{i\theta})$ is always different from zero, the above inequality holds for all (x, ξ') for some constant $C > 0$. We conclude that there exist $C > 0$ and $D > 0$ such that

$$\left| \left(\partial_x^\beta \partial_\xi^\alpha \frac{1}{a} \right) (x, \xi', \tilde{R} \langle \xi' \rangle e^{i\theta}) \right| \leq CD^{|\alpha|+|\beta|} \alpha! (\beta!)^\nu \langle x \rangle^{-m_2-|\beta|} \langle \xi' \rangle^{-m_1-|\alpha|}, \quad \forall (x, \xi') \in \mathbb{R}^n \times \mathbb{R}^{n-1}.$$

If $b \in S_{\mu\nu}^{-m_1, -m_2}(\mathbb{R}^n \times \mathbb{R}^n)$, $\mu > 1$, is a parametrix of a and $\chi \in G^{\min\{\mu, \nu\}}(\mathbb{R}^n \times \mathbb{R}^n)$ is a function that is zero, if $|(x, \xi)| \leq R$, and equal to 1, if $|(x, \xi)| \geq 2R$, then $b \sim \sum_{j=0}^{\infty} b_{-m_1-j, -m_2-j}$, where

$$\begin{aligned} b_{-m_1, -m_2} &= \chi \frac{1}{a}, \\ b_{-m_1-j, -m_2-j} &= \sum_{k+|\alpha|=j, k < j} \frac{1}{\alpha!} (D_\xi^\alpha b_{-m_1-k, -m_2-k}) (\partial_x^\alpha a) b_{-m_1, -m_2}, \quad j \geq 1. \end{aligned}$$

Using the above estimates, we conclude that b must satisfy the following conditions, which we will call assumption (A):

Definition 26. We say that a symbol $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$, $m_1 \in \mathbb{Z}$ and $m_2 \in \mathbb{R}$, satisfies the assumption (A) if there are rational functions of ξ , $(x, \xi) \mapsto a_{m_1-j, m_2-j}(x, \xi)$, and constants $C > 0$, $D > 0$, $B > 0$ and $r > 0$, with $B > r > 0$, such that

1) For all $|(x, \xi)| \geq B$, the following holds:

$$\left| (\partial_x^\beta \partial_\xi^\alpha a_{m_1-j, m_2-j})(x, \xi) \right| \leq CD^{|\alpha|+|\beta|+2j} (j! \beta!)^\nu \alpha! \langle x \rangle^{m_2-j-|\beta|} \langle \xi \rangle^{m_1-j-|\alpha|}.$$

2) If z_0 is the pole of the function $z \in \mathbb{C} \mapsto a_{m_1-j, m_2-j}(x, \xi', z)$, then $|z_0| \leq r \langle \xi' \rangle$. If this pole is real, $z_0 \in \mathbb{R}$, then $|(x, \xi', z_0)| \leq B$. Moreover, for all $(x, \xi') \in \mathbb{R}^n \times \mathbb{R}^{n-1}$:

$$\left| (\partial_x^\beta \partial_\xi^\alpha a_{m_1-j, m_2-j})(x, \xi', B \langle \xi' \rangle e^{i\theta}) \right| \leq CD^{|\alpha|+|\beta|+2j} (j! \beta!)^\nu \alpha! \langle x \rangle^{m_2-j-|\beta|} \langle \xi' \rangle^{m_1-j-|\alpha|}.$$

3) For each $M \in \mathbb{N}_0$ and $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\max\{\langle x \rangle, \langle \xi \rangle\} \geq BM^{\mu+\nu-1}$, we define the function

$$r_{m_1-M, m_2-M}(x, \xi) := a(x, \xi) - \sum_{j=0}^{M-1} a_{m_1-j, m_2-j}(x, \xi).$$

These functions satisfy, for $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\max\{\langle x \rangle, \langle \xi \rangle\} \geq BM^{\mu+\nu-1}$, the estimate:

$$\left| \partial_x^\beta \partial_\xi^\alpha r_{m_1-M, m_2-M}(x, \xi) \right| \leq CD^{|\alpha|+|\beta|+2M} (\alpha!)^\mu (\beta!)^\nu (M!)^{\mu+\nu-1} \langle x \rangle^{m_2-M-|\beta|} \langle \xi \rangle^{m_1-M-|\alpha|}.$$

In particular, let $\chi \in G^{\min\{\mu, \nu\}}(\mathbb{R}^n \times \mathbb{R}^n)$ be a function that is zero, if $|(x, \xi)| \leq B$, and equal to 1, if $|(x, \xi)| \geq 2B$, then $\sum_{j=0}^\infty \chi a_{m_1-j, m_2-j} \in FS_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ and $a \sim \sum_{j=0}^\infty \chi a_{m_1-j, m_2-j}$.

Remark 27. If $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ and $b \in S_{\mu'\nu'}^{m'_1, m'_2}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy our assumptions, then $op(a)op(b) = op(t) + R$, where R is a θ -regularizing operator for $\theta \geq \mu + \nu - 1$ and $t \in S_{\mu\nu}^{m_1+m'_1, m_2+m'_2}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies our assumptions. This follows from the fact that if $a \sim \sum_{j=0}^\infty a_{m_1-j, m_2-j}$ and $b \sim \sum_{j=0}^\infty b_{m'_1-j, m'_2-j}$, then

$$t \sim \sum_{l=0}^\infty \left[\sum_{j+k+|\alpha|=l} \frac{1}{\alpha!} (\partial_\xi^\alpha a_{m_1-j, m_2-j}) (D_x^\alpha b_{m'_1-k, m'_2-k}) \right].$$

There are some important consequences of the assumption (A). Let us define three paths in \mathbb{C} . The first one is defined as

$$\Gamma_{\xi'} := \{B \langle \xi' \rangle e^{i\theta}; 0 \leq \theta \leq \pi\}.$$

The second one is defined as

$$\Gamma_{\xi', M} := \gamma_1 \cup \Gamma_{\xi'} \cup \gamma_2,$$

where γ_1 is the real line that starts at $BM^{\mu+\nu-1}$ and ends at $B \langle \xi' \rangle$ and γ_2 is the real line that starts at $-B \langle \xi' \rangle$ and ends at $-BM^{\mu+\nu-1}$.

The third path we are going to use is

$$\Gamma_{(\xi'), M} := \gamma_3 \cup \Gamma_{\xi'} \cup \gamma_4,$$

where γ_3 is the real line that starts at $\sqrt{B^2 M^{2(\mu+\nu-1)} - \langle \xi' \rangle^2}$, if it is a real number, or at 0, if it is not real, and ends at $B \langle \xi' \rangle$. The curve γ_4 is the real line that starts at $-B \langle \xi' \rangle$ and ends at $-\sqrt{B^2 M^{2(\mu+\nu-1)} - \langle \xi' \rangle^2}$, if it is a real number, or at 0, if it is not real.

Now let us suppose that $v \in \mathcal{S}'(\mathbb{R}^n)$ is a distribution such that $(\xi', \xi_n) \mapsto \hat{v}(\xi)$ extends to a continuous function from $\mathbb{R}^{n-1} \times \mathbb{H}$ to \mathbb{C} and $\xi_n \in \mathring{\mathbb{H}} \mapsto \hat{v}(\xi', \xi_n)$ is holomorphic for each ξ' fixed. Suppose that $|\xi'^{\alpha'} \hat{v}(\xi', \xi_n)|$ is bounded, for all $\alpha' \in \mathbb{N}_0^{n-1}$ and all $(\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{H}$. This is the case of $v = e^- u$, where $u \in \mathcal{S}(\mathbb{R}_-^n)$, and of $v = u \otimes \delta$, where $u \in \mathcal{S}(\mathbb{R}^{n-1})$ and δ is the delta distribution in x_n . If a is a symbol that satisfies the assumption (A), we can use Cauchy Theorem to show that, for $M \geq m_1 + n + 1$ and $x_n > 0$:

$$\begin{aligned} op(a)v(x) = & \frac{1}{(2\pi)^n} \int_{\langle \xi \rangle \leq BM^{\mu+\nu-1}} e^{ix\xi} a(x, \xi) \hat{v}(\xi) d\xi + \\ & \sum_{j=0}^{M-1} \frac{1}{(2\pi)^n} \int \left(\int_{\Gamma_{(\xi'), M}} e^{ix\xi} a_{m_1-j, m_2-j}(x, \xi', \xi_n) \hat{v}(\xi', \xi_n) d\xi_n \right) d\xi' + \\ & \frac{1}{(2\pi)^n} \int_{\langle \xi \rangle \geq BM^{\mu+\nu-1}} e^{ix\xi} r_{m_1-M, m_2-M}(x, \xi) \hat{v}(\xi) d\xi. \end{aligned}$$

and

$$\begin{aligned} op(a)v(x) = & \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \left(\int_{|\xi_n| \leq BM^{\mu+\nu-1}} e^{ix\xi} a(x, \xi', \xi_n) \hat{v}(\xi', \xi_n) d\xi_n \right) d\xi' + \\ & \sum_{j=0}^{M-1} \frac{1}{(2\pi)^n} \int \left(\int_{\Gamma_{\xi', M}} e^{ix\xi} a_{m_1-j, m_2-j}(x, \xi', \xi_n) \hat{v}(\xi', \xi_n) d\xi_n \right) d\xi' + \\ & \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \left(\int_{|\xi_n| \geq BM^{\mu+\nu-1}} e^{ix\xi} r_{m_1-M, m_2-M}(x, \xi) \hat{v}(\xi', \xi_n) d\xi_n \right) d\xi'. \end{aligned}$$

This can be used to prove the following proposition:

Proposition 28. *Let $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol that satisfies the assumption (A). If $u \in \mathcal{S}(\mathbb{R}^{n-1})$, then the following operator*

$$A^{kj}(u) = \lim_{x_n \rightarrow 0^+} D_{x_n}^k op(a)(u \otimes \delta^{(j)})$$

is well defined. Moreover $A^{kj} = op(a^{kj})$, where $a^{kj} \in S_{\mu\nu}^{m_1+j+k+1, m_2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$.

Proof. We only need to prove it for $k = j = 0$, as $D_{x_n}^k op(a) D_{x_n}^j$ is also a pseudo-differential operator whose symbol satisfies assumption (A). The proof is similar to [8, Lemma 1]. Let us choose and fix an integer $M \geq \max\{1, m_1 + n + 1\}$. Using the remark about the consequences of assumption (A) and the path $\Gamma_{\xi', M}$, we can easily take the limit in x_n to obtain:

$$\lim_{x_n \rightarrow 0^+} Au(x) = Cu(x') + R_M u(x') + \sum_{j=0}^{M-1} C_j u(x'),$$

where

$$\begin{aligned} Cu(x') &= \frac{1}{(2\pi)^{n-1}} \int e^{ix'\xi'} \left(\frac{1}{2\pi} \int_{|\xi_n| \leq BM^{\mu+\nu-1}} a(x', 0, \xi', \xi_n) d\xi_n \right) \hat{u}(\xi') d\xi', \\ R_M u(x') &= \frac{1}{(2\pi)^{n-1}} \int e^{ix'\xi'} \left(\frac{1}{2\pi} \int_{|\xi_n| \geq BM^{\mu+\nu-1}} r_{m_1-M, m_2-M}(x', 0, \xi', \xi_n) d\xi_n \right) \hat{u}(\xi') d\xi', \\ C_j u(x') &= \frac{1}{(2\pi)^{n-1}} \int e^{ix'\xi'} \left(\frac{1}{2\pi} \int_{\Gamma_{\xi', M}} a_{m_1-j, m_2-j}(x', 0, \xi', \xi_n) d\xi_n \right) \hat{u}(\xi') d\xi'. \end{aligned}$$

Now let us note that

$$\begin{aligned} & \left| \partial_{x'}^\beta \partial_{\xi'}^\alpha \left(\int_{|\xi_n| \leq BM^{\mu+\nu-1}} a(x', 0, \xi', \xi_n) d\xi_n \right) \right| \leq \\ & CD^{|\alpha|+|\beta|} (\alpha!)^\mu (\beta!)^\nu \langle x' \rangle^{m_2-|\beta|} \int_{|\xi_n| \leq BM^{\mu+\nu-1}} \langle (\xi', \xi_n) \rangle^{m_1-|\alpha|} d\xi_n \leq \\ & C_1 D_1^{|\alpha|+|\beta|} (\alpha!)^\mu (\beta!)^\nu \langle x' \rangle^{m_2-|\beta|} \langle \xi' \rangle^{m_1-|\alpha|}, \end{aligned}$$

where C_1 and D_1 are constants that depend on C , D , M , μ , ν and B , but not on α or β .

If $k < -n-1$, then $\int_{\mathbb{R}} \langle (\xi', \xi_n) \rangle^k d\xi_n = \langle \xi' \rangle^{k+1} \int \langle \xi_n \rangle^k d\xi_n$. Hence

$$\begin{aligned} & \left| \partial_{x'}^\beta \partial_{\xi'}^\alpha \left(\int_{|\xi_n| \geq BM^{\mu+\nu-1}} r_{m_1-M, m_2-M}(x', 0, \xi', \xi_n) d\xi_n \right) \right| \leq \\ & CD^{|\alpha|+|\beta|+2M} (M!)^{\mu+\nu-1} (\alpha!)^\mu (\beta!)^\nu \langle x' \rangle^{m_2-M-|\beta|} \int_{\mathbb{R}} \langle (\xi', \xi_n) \rangle^{m_1-M-|\alpha|} d\xi_n \leq \\ & C \left(\int_{\mathbb{R}} \langle \xi_n \rangle^{-n-1} d\xi_n \right) D^{|\alpha|+|\beta|+2M} (M!)^{\mu+\nu-1} (\alpha!)^\mu (\beta!)^\nu \langle x' \rangle^{m_2-M-|\beta|} \langle \xi' \rangle^{m_1+1-M-|\alpha|}. \end{aligned}$$

In order to study the integral in $\Gamma_{\xi', M}$, we first study in $\Gamma_{\xi'}$. Using item 2 of assumption (A) we obtain

$$\begin{aligned} & \left| \partial_{x'}^\beta \partial_{\xi'}^\alpha \left(\int_{\Gamma_{\xi'}} a_{m_1-j, m_2-j}(x', 0, \xi', \xi_n) d\xi_n \right) \right| \leq \\ & CD^{|\alpha|+|\beta|+2j} (j!)^\nu \alpha! (\beta!)^\nu \langle x' \rangle^{m_2-j-|\beta|} \left| \int_{\Gamma_{\xi'}} \langle \xi' \rangle^{m_1-j-|\alpha|} d\xi_n \right| \leq \\ & \pi BCD^{2M} (M!)^\nu D^{|\alpha|+|\beta|} \alpha! (\beta!)^\nu \langle x' \rangle^{m_2-j-|\beta|} \langle \xi' \rangle^{m_1+1-j-|\alpha|}. \end{aligned}$$

Finally, using item 1 of assumption (A), we obtain

$$\begin{aligned} & \left| \partial_{x'}^\beta \partial_{\xi'}^\alpha \left(\int_{\gamma_1 \cup \gamma_2} a_{m_1-j, m_2-j}(x', 0, \xi', \xi_n) d\xi_n \right) \right| \leq \\ & CD^{|\alpha|+|\beta|+2j} (j!)^\nu \alpha! (\beta!)^\nu \langle x' \rangle^{m_2-j-|\beta|} \int_{|\xi_n| \leq \max\{B\langle \xi' \rangle, BM^{\mu+\nu-1}\}} \langle \xi \rangle^{m_1-j-|\alpha|} d\xi_n \leq \\ & C_1 D_1^{|\alpha|+|\beta|} \alpha! (\beta!)^\nu \langle x' \rangle^{m_2-j-|\beta|} \langle \xi' \rangle^{m_1+1-j-|\alpha|}, \end{aligned}$$

where C_1 does not depend on α or β . It only depends on C , D , M , μ , ν and B . □

Now let us prove what we call the Gelfand-Shilov transmission property.

Proposition 29. *Let $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol that satisfies the assumption (A). Then $r^+op(a)e^+$ is a map from $\mathcal{S}_\theta^\theta(\mathbb{R}_+^n)$ to $\mathcal{S}_\theta^\theta(\mathbb{R}_+^n)$, where $\theta \geq \mu + \nu - 1$. This property will be called Gelfand-Shilov transmission property.*

The proof is based on the one given by J. Chazarain and A. Piriou [4, Chapter 5, Section 2].

Lemma 30. *Let $u \in \mathcal{S}_\theta^\theta(\mathbb{R}_-^n)$. Then*

1) *The function $\xi \in \mathbb{R}_-^n \rightarrow \widehat{e^-u}(\xi) \in \mathbb{C}$ extends to a continuous function $(\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{H} \mapsto \widehat{e^-u}(\xi) \in \mathbb{C}$ such that $\xi_n \in \mathbb{H} \mapsto \widehat{e^-u}(\xi', \xi_n) \in \mathbb{C}$ is a holomorphic function for each ξ' fixed.*

2) *There exist constants $E > 0$ and $F > 0$ such that, for all $l \in \mathbb{N}_0$, the following estimate holds:*

$$|D_\xi^\sigma(\widehat{e^-u})(\xi)| \leq EF^{l+|\sigma|} (l!)^\theta (\sigma!)^\theta \langle \xi' \rangle^{-l},$$

for every $\xi := (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{H}$.

Proof. 1) By definition $\widehat{e^-u}(\xi) := \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^0 e^{-ix\xi} u(x) dx_n \right) dx'$. Hence if $Im(\xi_n) \geq 0$ and $x_n \leq 0$, then $Re(-ix_n \xi_n) \leq 0$. Therefore the integral is well defined and it is straightforward to see that it defines an analytic function of ξ_n in the upper half-plane for each ξ' fixed.

2) Using integration by parts, we obtain

$$\left| \xi'^\gamma D_\xi^\sigma \int_{\mathbb{R}_-^n} e^{-ix\xi} u(x) dx \right| \leq \int_{\mathbb{R}_-^n} (1 + |x|^2)^{-n} \left| (1 + |x|^2)^n D_{x'}^\gamma (x^\sigma u)(x) \right| dx \leq \left(\int_{\mathbb{R}_-^n} (1 + |x|^2)^{-n} dx \right) (n!)^\theta C D^{|\gamma|+|\sigma|+n} (\sigma! \gamma!)^\theta,$$

for some constants $C > 0$ and $D > 0$. The result follows from the above estimate. \square

Proposition 31. *Let $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol that satisfies the assumption (A) and $u \in \mathcal{S}_\theta^\theta(\mathbb{R}_-^n)$, where $\theta \geq \mu + \nu - 1$. Hence $r^+op(a)e^-(u)$ belongs to $\mathcal{S}_\theta^\theta(\mathbb{R}_+^n)$.*

Proof. Along the proof C, D, C_1, D_1, \dots indicate constants that do not depend on the multi-indices α and β (and so neither on γ nor on σ as defined below). Sometimes we use the same letters to indicate different constants only to avoid a too messy notation.

We start using integration by parts to obtain

$$x^\alpha \partial_x^\beta \int e^{ix\xi} a(x, \xi) \widehat{e^-u}(\xi) d\xi = \sum_{\gamma \leq \alpha} \sum_{\sigma \leq \beta} \binom{\alpha}{\gamma} \binom{\beta}{\sigma} (-1)^{|\alpha|} \int e^{ix\xi} D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} a)(x, \xi)) \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right)(\xi) d\xi.$$

Let us study the term $\int e^{ix\xi} D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} a)(x, \xi)) \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right)(\xi) d\xi$. We define

$$M := \max \{0, |\sigma| + m_1 + n + 1\}.$$

Using the remark about the consequences of assumption (A), we know that

$$(3.2) \quad \int e^{ix\xi} D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} a)(x, \xi)) \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right)(\xi) d\xi =$$

$$\begin{aligned}
& \int_{\langle \xi \rangle \leq BM^{\mu+\nu-1}} e^{ix\xi} D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} a)(x, \xi)) \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right) (\xi) d\xi + \\
& \sum_{j=0}^{M-1} \int e^{ix'\xi'} \left(\int_{\Gamma_{(\xi'), M}} e^{ix_n \xi_n} D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} a_{m_1-j, m_2-j})(x, \xi)) \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right) (\xi) d\xi_n \right) d\xi' + \\
& \int_{\langle \xi \rangle \geq BM^{\mu+\nu-1}} e^{ix\xi} D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} r_{m_1-M, m_2-M})(x, \xi)) \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right) (\xi) d\xi.
\end{aligned}$$

The second line of Equation 3.2 is such that

$$\begin{aligned}
& \left| \int_{\langle \xi \rangle \leq BM^{\mu+\nu-1}} e^{ix\xi} D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} a)(x, \xi)) \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right) (\xi) d\xi \right| \leq \\
& CD^{|\beta|+|\gamma|} (\gamma!)^\mu (\beta - \sigma)!^\nu \langle x \rangle^{m_2-|\beta-\sigma|} EF^{|\alpha-\gamma|} (\alpha - \gamma)!^\theta \int_{\langle \xi \rangle \leq BM^{\mu+\nu-1}} \langle \xi \rangle^{m_1-|\gamma|+|\sigma|} d\xi \leq \\
& C_1 D_1^{|\beta|+|\alpha|} (\gamma!)^\mu (\beta - \sigma)!^\nu \langle x \rangle^{m_2-|\beta-\sigma|} (\alpha - \gamma)!^\theta (BM^{\mu+\nu-1})^{|m_1|+|\sigma|+n} \leq C_2 D_2^{|\beta|+|\alpha|} (\alpha!)^\theta (\beta!)^\theta \langle x \rangle^{m_2}.
\end{aligned}$$

We have used that $(BM^{\mu+\nu-1})^{|m_1|+|\sigma|+n} \leq CD^{|\sigma|} (\sigma!)^{\mu+\nu-1}$ and that

$$|D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} a)(x, \xi))| \leq CD^{|\gamma|+|\beta|} (\gamma!)^\mu (\beta - \sigma)!^\nu \langle x \rangle^{m_2-|\beta-\sigma|} \langle \xi \rangle^{m_1+|\sigma-\gamma|}.$$

Let us now study the term in the fourth line of Equation 3.2. We note that, in the case $M = 0$, this is the only term that appears in the right hand side of Equation 3.2 and $r_{m_1-M, m_2-M} = a$ in that situation.

For $M \geq 0$, we have

$$\begin{aligned}
& \left| \int_{\langle \xi \rangle \geq BM^{\mu+\nu-1}} e^{ix\xi} D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} r_{m_1-M, m_2-M})(x, \xi)) \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right) (\xi) d\xi \right| \leq \\
& CD^{|\beta|+|\gamma|} (\beta - \sigma)!^\nu (\gamma!)^\mu \langle x \rangle^{m_2-M-|\beta-\sigma|} EF^{|\alpha|-|\gamma|} (\alpha - \gamma)!^\theta \int_{\mathbb{R}^n} \langle \xi \rangle^{m_1-M+|\sigma|-|\gamma|} d\xi \leq \\
& C_1 D_1^{|\alpha|+|\beta|} (\alpha!)^\theta (\beta - \sigma)!^\nu \langle x \rangle^{m_2},
\end{aligned}$$

where we used that $m_1 - M + |\sigma| - |\gamma| \leq -n - 1$.

Finally we study the term of the third line of Equation 3.2. Without loss of generality, we can suppose that $M = |\sigma| + m_1 + n + 1 > 0$ as $0 \leq j < M$.

First we study the integral in $\Gamma_{\xi'}$:

$$\begin{aligned}
& \left| \int e^{ix'\xi'} \left(\int_{\Gamma_{\xi'}} e^{ix_n \xi_n} D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} a_{m_1-j, m_2-j})(x, \xi)) \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right) (\xi) d\xi_n \right) d\xi' \right| \leq \\
& CD^{|\beta|+|\gamma|} (\beta - \sigma)!^\nu \gamma! (j!)^\nu \langle x \rangle^{m_2-j-|\beta|+|\sigma|} \int_{\mathbb{R}^{n-1}} \left(\int_{\Gamma_{\xi'}} \langle \xi' \rangle^{m_1-j+|\sigma|-|\gamma|} \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right) (\xi) d\xi_n \right) d\xi'.
\end{aligned}$$

In the above expressions, we have used item 2 of assumption (A). Using the estimate of Lemma 30 with $l = m_1 + n + 1 + |\sigma| - j$, we conclude that the above expression is smaller than $C_1 D_1^{|\beta|+|\alpha|} (\alpha!)^\theta (\beta!)^\theta \langle x \rangle^{m_2}$.

Finally the integral in $\gamma_3 \cup \gamma_4$ can be evaluated as follows

$$(3.3) \quad \left| \int e^{ix'\xi'} \left(\int_{\gamma_3 \cup \gamma_4} e^{ix_n \xi_n} D_\xi^\gamma ((i\xi)^\sigma (\partial_x^{\beta-\sigma} a_{m_1-j, m_2-j})(x, \xi)) \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right)(\xi) d\xi_n \right) d\xi' \right| \leq$$

$$CD^{|\beta|+|\gamma|} (\beta - \sigma)!^\nu \gamma! (j!)^\nu \langle x \rangle^{m_2-j-|\beta|+|\sigma|}$$

$$\left| \int \left(\int_{|\xi_n| \leq \max\{B\langle \xi' \rangle, \sqrt{B^2 M^{2(\mu+\nu-1)} - \langle \xi' \rangle^2}\}} \langle \xi \rangle^{m_1-j+|\sigma|-|\gamma|} \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right)(\xi) d\xi_n \right) d\xi' \right|.$$

If $\max\left\{B\langle \xi' \rangle, \sqrt{B^2 M^{2(\mu+\nu-1)} - \langle \xi' \rangle^2}\right\} = B\langle \xi' \rangle$, then $\langle \xi \rangle^{m_1-j+|\sigma|-|\gamma|} \leq CD^{m_1-j+|\sigma|-|\gamma|} \langle \xi' \rangle^{m_1-j+|\sigma|-|\gamma|}$ in the integrand, where C and D are constants that depend on B . Using Lemma 30 with $l := m_1 + n + 1 + |\sigma| - j$, we conclude that the above integral is smaller than $C_1 D_1^{|\beta|+|\alpha|} (\alpha!)^\theta (\beta!)^\theta \langle x \rangle^{m_2}$.

If $\max\left\{B\langle \xi' \rangle, \sqrt{B^2 M^{2(\mu+\nu-1)} - \langle \xi' \rangle^2}\right\} = \sqrt{B^2 M^{2(\mu+\nu-1)} - \langle \xi' \rangle^2}$, then $\langle \xi \rangle \leq BM^{\mu+\nu-1}$ in the integrand and

$$\left(\int_{\langle \xi \rangle \leq BM^{\mu+\nu-1}} \langle \xi \rangle^{m_1-j+|\sigma|-|\gamma|} \left(D_\xi^{\alpha-\gamma} \widehat{e^-u} \right)(\xi) d\xi \right) \leq$$

$$\begin{cases} EF^{|\alpha-\gamma|} (\alpha - \gamma)!^\theta \left(\int_{|\xi| \leq 1} d\xi \right) (BM^{\mu+\nu-1})^{n+m_1-j+|\sigma|-|\gamma|}, & \text{if } m_1 - j + |\sigma| > |\gamma| \\ EF^{|\alpha-\gamma|} (\alpha - \gamma)!^\theta \left(\int_{|\xi| \leq 1} d\xi \right) B^n M^{n(\mu+\nu-1)}, & \text{if } m_1 - j + |\sigma| \leq |\gamma| \end{cases}.$$

In order to conclude that the expression of Equation 3.3 is smaller than $C_1 D_1^{|\beta|+|\alpha|} (\alpha!)^\theta (\beta!)^\theta \langle x \rangle^{m_2}$, we use in the first situation, when $m_1 - j + |\sigma| > |\gamma|$, that $(j!)^\nu M^{-j(\mu+\nu-1)} \leq 1$ and $M^{(\mu+\nu-1)(n+m_1+|\sigma|-|\gamma|)} \leq CD^{|\sigma|} (\sigma!)^{\mu+\nu-1}$. In the second situation, $(j!)^\nu \leq CD^{|\sigma|} (\sigma!)^\nu$ is used. \square

The proof of Proposition 29 now follows easily:

Proof. (of Proposition 29) Let $f \in \mathcal{S}_\theta^\theta(\mathbb{R}_+^n)$. Let us choose $\tilde{f} \in \mathcal{S}_\theta^\theta(\mathbb{R}^n)$ such that $r^+(\tilde{f}) = f$, which exists according to Theorem 3. Let $h \in \mathcal{S}_\theta^\theta(\mathbb{R}_-^n)$ be defined as $h := r^-(\tilde{f})$. Hence

$$r^+ op(a) e^+(f) = r^+ op(a) \left(\tilde{f} - e^-(h) \right) = r^+ op(a) \left(\tilde{f} \right) - r^+ op(a) e^-(h).$$

By Proposition 13, we know that $op(a) \left(\tilde{f} \right) \in \mathcal{S}_\theta^\theta(\mathbb{R}^n)$. Hence $r^+ op(a) \left(\tilde{f} \right) \in \mathcal{S}_\theta^\theta(\mathbb{R}_+^n)$. We conclude the proof using Proposition 31 to obtain $r^+ op(a) e^-(h) \in \mathcal{S}_\theta^\theta(\mathbb{R}_+^n)$. \square

We conclude studying Poisson operators, similar to the ones in L. B. de Monvel [17], to obtain the following result:

Proposition 32. *Let $a \in S_{\mu\nu}^{m_1, m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol that satisfies the assumption (A). If $\theta \geq \mu + \nu - 1$ and $v \in \mathcal{S}_\theta^\theta(\mathbb{R}^{n-1})$, then*

$$x' \in \mathbb{R}^{n-1} \mapsto r^+ (op(a) (v(x') \otimes \delta(x_n))) \in \mathcal{S}_\theta^\theta(\mathbb{R}_+^n).$$

Proof. We just have to note that for $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$op(a) D_n (e^+(u)) = op(a) e^+ (D_n u) + \frac{1}{i} op(a) (u(x', 0) \otimes \delta(x_n)).$$

Let us choose a function $u \in \mathcal{S}_\theta^\theta(\mathbb{R}^n)$ such that $u(x', 0) = v(x')$ for $x' \in \mathbb{R}^{n-1}$. For instance, $u(x) = v(x')e^{-x_n^2}$. Using the Gelfand-Shilov transmission property, we conclude that $r^+(op(a)D_n(e^+(u)))$ and $r^+(op(a)(e^+(D_n u)))$ belong to $\mathcal{S}_\theta^\theta(\mathbb{R}_+^n)$. By the above expression, the same must hold for the expression $r^+(op(a)(u(x', 0) \otimes \delta(x_n)))$. \square

Combining all the previous results, we now prove Theorem 25.

Proof. (of the Main Theorem on the half-space, Theorem 25).

We first choose $\mu > 1$, such that $\nu + \mu - 1 < \theta$ and write $P(x, D)$ as $P(x, D) = \sum_{j=0}^{m_1} P_j(x, D_{x'}) D_{x_n}^j$, where $P_j(x, D_{x'})$ is a differential operator in $D_{x'}$ of order $\leq m_1 - j$.

We then define the function $\tilde{P} : \mathcal{S}(\mathbb{R}^{n-1})^{\oplus m_1} \rightarrow \mathcal{S}'(\mathbb{R}^n)$ as

$$\tilde{P}(v_0, \dots, v_{m_1-1}) = \frac{1}{i} \sum_{l=0}^{m_1-1} \sum_{j=0}^{m_1-l-1} P_{j+l+1}(x', 0, D') v_l \otimes \delta^{(j)}.$$

Hence if $\gamma : \mathcal{S}(\mathbb{R}_+^n) \rightarrow \mathcal{S}(\mathbb{R}^{n-1})^{\oplus m_1}$ is the function given by $\gamma(u) = (\gamma_0(u), \dots, \gamma_{m_1-1}(u))$, we conclude that, if $u \in \mathcal{S}(\mathbb{R}_+^n)$, then

$$(3.4) \quad P(e^+u) = e^+P(u) + \frac{1}{i} \sum_{l=0}^{m_1-1} \sum_{j=0}^{m_1-l-1} P_{j+l+1}(x', 0, D') \gamma_l(u) \otimes \delta^{(j)} = e^+P(u) + \tilde{P}\gamma(u).$$

Now let u be the solution of Equation 3.1. We know that $u \in \mathcal{S}(\mathbb{R}_+^n) = \cap_{(s,t) \in \mathbb{R}^2} H^{s,t}(\mathbb{R}_+^n)$, due to the SG-ellipticity of the problem and the fact that $f \in \mathcal{S}(\mathbb{R}_+^n)$ and $g_j \in \mathcal{S}(\mathbb{R}^{n-1})$ for all j , see [9]. Let $b \in S_{\mu\nu}^{-m_1, -m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ be a parametrix of a . If we apply r^+Q , where $Q = op(b)$, to both sides of Equation 3.4 and use that $QP = I + R$, where R is a θ -regularizing operator, we obtain:

$$(3.5) \quad u = r^+Qe^+(f) + r^+Q\tilde{P}\gamma(u) - r^+R(e^+u).$$

Applying γ to the above equation, we conclude that $\gamma(u)$ must satisfy

$$(3.6) \quad \begin{aligned} (I - \gamma Q \tilde{P}) \gamma(u) &= \gamma(Qe^+(f)) - \gamma(R(e^+u)) \\ B\gamma(u) &= g \end{aligned},$$

where $g = (g_1, \dots, g_r)$ and $(B\gamma(u))_j = \sum_{k=0}^{m_1-1} B_{j,k}(x', D') \gamma_k(u)$.

Explicitly, this means that

$$\gamma(u) - \frac{1}{i} \sum_{l=0}^{m_1-1} \sum_{j=0}^{m_1-l-1} \gamma(Q(P_{j+l+1}(x', 0, D') \gamma_l(u) \otimes \delta^{(j)})) = \gamma(Qe^+(f)) - \gamma(R(e^+u)) \\ \sum_{k=0}^{m_1j} B_{j,k}(x', D') \gamma_k(u) = g_j, \quad j = 1, \dots, r.$$

According to Proposition 28, the function $Q^{kj} : \mathcal{S}(\mathbb{R}^{n-1}) \rightarrow \mathcal{S}(\mathbb{R}^{n-1})$ defined as $Q^{kj}(v) := \gamma_k(Q(v \otimes \delta^{(j)}))$ defines a pseudo-differential operator with symbol in $S_{\mu\nu}^{-m_1+j+k+1, -m_2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. We now define the following functions

$$U_j = \langle D' \rangle^{-j} \gamma_j(u), \quad F^j = \langle D' \rangle^{-j} \gamma_j(Qe^+(f) - R(e^+u)) \quad \text{and} \quad G^j = \langle x' \rangle^{-m_{2j}} \langle D' \rangle^{-m_{1j}} g_j,$$

and operators

$$\overline{Q} = \left(\frac{1}{i} \sum_{j=0}^{m_1-l-1} \langle D' \rangle^{-k} Q^{kj} P_{j+l+1}(x', D') \langle D' \rangle^l \right)_{k,l} \quad \text{and} \quad \overline{B} = \left(\langle x' \rangle^{-m_{2k}} \langle D' \rangle^{-m_{1k}} B_{k,l}(x', D) \langle D' \rangle^l \right)_{k,l}.$$

Using the Gelfand-Shilov Transmission Property, Proposition 29, and the fact that R is θ -regularizing operator, we conclude that $F^j \in \mathcal{S}_\theta^\theta(\mathbb{R}^{n-1})$. As $g_j \in \mathcal{S}_\theta^\theta(\mathbb{R}^{n-1})$, using Proposition 13, we conclude that $G_j \in \mathcal{S}_\theta^\theta(\mathbb{R}^{n-1})$, for all j .

Equation 3.6 is equivalent to

$$\begin{pmatrix} I - \overline{Q} \\ \overline{B} \end{pmatrix} \begin{pmatrix} U_0 \\ \vdots \\ U_{m_1-1} \end{pmatrix} = \begin{pmatrix} F_0 \\ \vdots \\ F_{m_1-1} \\ G_0 \\ \vdots \\ G_{r-1} \end{pmatrix}.$$

The operator $\begin{pmatrix} I - \overline{Q} \\ \overline{B} \end{pmatrix}$ is a pseudo-differential operator, whose symbol belongs to

$$S_{\mu\nu}^{0,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{B}(\mathbb{C}^{m_1}, \mathbb{C}^{\frac{3}{2}m_1})).$$

As a consequence of SG-Lopatinski-Shapiro condition [8, Lemma 2], this operator is a left elliptic pseudo-differential operator. Therefore by Theorem 18, $U_j \in \mathcal{S}_\theta^\theta(\mathbb{R}^{n-1})$ for all j and $\gamma_j(u) = \langle D' \rangle^j U_j \in \mathcal{S}_\theta^\theta(\mathbb{R}^{n-1})$. Using the Gelfand-Shilov Transmission Property, Proposition 29, we conclude that $r^+ Q e^+(f) \in \mathcal{S}_\theta^\theta(\mathbb{R}_+^n)$. The properties of the Poisson operator, Proposition 32, imply that $r^+ Q \tilde{P} \gamma(u) \in \mathcal{S}_\theta^\theta(\mathbb{R}^n)$. As R is a θ -regularizing operator, the result follows then from Equation 3.5. \square

We note that, in the previous proof, we only need the left ellipticity of $\begin{pmatrix} I - \overline{Q} \\ \overline{B} \end{pmatrix}$. Hence the result holds for even more general boundary value problems operators than just the ones that satisfy the SG-Lopatinski-Shapiro condition.

ACKNOWLEDGEMENTS

The author would like to thank Professor Jorge G. Hounie and Professor Elmar Schrohe for fruitful discussions. We would also like to thank Professors G. Hoepfner, R. F. Barostichi, J. R. Santos Filho from UFSCar and L. Rodino from Torino for suggesting references.

REFERENCES

1. A. Calderón. *Boundary value problems for elliptic equations*. Outlines of the Joint Soviet-American Sympos. on PDE's (Novosibirsk), 303-304 (1963).
2. M. Capriello, T. Gramchev and L. Rodino. *Gelfand-Shilov Spaces, Pseudo-differential Operators and Localization Operators*. Modern Trends in Pseudo-Differential Operators. Birkhäuser Basel, 172: 297-312 (2006).

3. M. Capiello and L. Rodino. *SG-pseudodifferential operators and Gelfand-Shilov spaces*. Rocky Mountain J. Math., 36:1117-1148, (2006).
4. J. Chazarain and A. Piriou. *Introduction to the Theory of Linear Partial Differential Equations*. North-Holland, (1982).
5. H. O. Cordes. *A Global parametrix for pseudodifferential operators over \mathbb{R}^n , with applications*. Preprint, Universität Bonn, (1976).
6. H. O. Cordes and A. K. Erkip. *The N -th order elliptic boundary problem for noncompact boundaries*. Rocky Mountain J. Math., 10: 7-24 (1980).
7. G. A. Dzasasija. *Carleman's problem for functions of the Gevrey class*. Soviet Math Dokl, 3: 969-972 (1962).
8. A. K. Erkip. *The elliptic boundary problem on the half space*. Communications in P.D.E, 4: 537-554 (1979).
9. A. K. Erkip. *Normal Solvability of Boundary Value Problems in Half Space*. Lecture Notes in Mathematics, 1256: 123-134 (1987).
10. A. K. Erkip and E. Schrohe. *Normal Solvability of Elliptic Boundary Value Problems on Asymptotically Flat Manifolds*. Journal of Functional Analysis, 109.1: 22-51 (1992).
11. I. M. Gel'fand and G.E. Shilov. *Generalized Functions II*. Academic Press, (1968).
12. L. Hörmander. *Pseudo-differential Operators and Non-elliptic Boundary Problems*. Annals of Mathematics, 83: 129-209 (1966).
13. D. Kapanadze and B.-W. Schulze. *Boundary Value Problems on Manifolds with Exits to Infinity*. Rend. Sem. Mat. Univ. Pol. Torino, 58, 3: 301-360 (2000).
14. D. Kapanadze and B.W. Schulze. *Crack Theory and Edge Singularities*. Kluwer Academic Publishers, (2003).
15. J. L. Lions and E. Magenes. *Non-homogeneous Boundary Value Problems and Applications: Vol. 3*. Springer-Verlag, (1973).
16. R. Melrose. *Geometric Scattering Theory*. Cambridge University Press, (1995).
17. L. B. Monvel. *Boundary problems for pseudo-differential operators*. Acta mathematica, 126: 11-51 (1971).
18. F. Nicola and L. Rodino. *Global Pseudo-differential calculus on Euclidean spaces*. Birkhäuser, (2011).
19. C. Parenti. *Operatori pseudo-differenziali in \mathbb{R}^n e applicazioni*. Annali di matematica pura ed applicata, 93.1: 359-389 (1972).
20. R. T. Seeley. *Singular integrals and boundary value problems*. Amer. J. Math., 88: 781-809 (1966).
21. E. Schrohe. *Fréchet algebra techniques for boundary value problems on noncompact manifolds: Fredholm criteria and functional calculus via spectral invariance*. Math. Nachr., 199.1: 145-185, (1999).
22. B. W. Schulze. *Boundary Value Problems and Singular Pseudo-differential operators*. John Wiley, (1998).
23. J. T. Wloka, B. Rowley and B. Lawruk. *Boundary value problems for elliptic systems*. Cambridge University Press, (1995).

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